# Chapter 12 – 6b Optimization

**Example 1.** A paint manufacturer has a uniform annual demand for 16,000 cans of automobile primer. It costs $4 to store one can of paint for one year and $500 to set up the plant for production of the primer. How many times a year should the company produce this primer in order to minimize storage and setup costs?

Problems of this type are based on two assumptions (whether they are stated or not). One is that the demand is uniform (the same every business day). The other is that when the company produces the product, it produces the same quantity every time.

Let x represent the number of cans produced at a time and let y represent the number of production runs during the year.

$$xy=16,000 ⟹ y=\frac{16,000}{x} 1\leq x\leq 16,000$$

 Over the course of time between one production run and the next, the number of cans being stored per day is x/2. Consequently, over the course of the entire year, the number of cans being stored is also x/2 cans per day. Since the average daily storage rate is a constant, over the course of the entire year, the company is storing x/2 cans per year.

$Cost=500y+4∙\frac{x}{2}$

$$C\left(x\right)=500∙\frac{16,000}{x}+4∙\frac{x}{2}=\frac{8,000,000}{x}+2x 1\leq x\leq 16,000$$

$$C^{'}\left(x\right)=-\frac{8,000,000}{x^{2}}+2=0 ⟹ x=2,000 cans per run$$

$$C^{''}\left(x\right)=\frac{16,000,000}{x^{3}}>0 for x>0$$

$$y=\frac{16,000}{x}=\frac{16,000}{2,000}=8 production runs per year$$

$$C\left(2,000\right)=\frac{8,000,000}{2,000}+2∙2000=\$8,000$$

The minimum cost is $8,000 when paint is produced eight times a year in batches of 2,000 cans at a time.

**Example 2.** A commercial cherry grower estimates from past records that if 30 trees are planted per acre, each tree will yield an average of 50 pounds of cherries per season. If, for each additional tree planted per acre (up to 20), the average yield per tree is reduced by 1 pound, how many trees should be planted per acre to obtain the maximum yield per acre? What is the maximum yield per acre?

If $x$ is the number of trees per acre, the yield (in pounds) per tree is

$$p\left(x\right)=50-\left(x-30\right)=80-x for 30\leq x\leq 50$$

The yield (in pounds) per acre is given by

$$P\left(x\right)=x∙p\left(x\right)=80x-x^{2}$$

$P^{'}\left(x\right)=80-2x=0$ at $x=40$

$P^{''}\left(x\right)=-2$ means the yield function is always concave down.

The cherry grower should plant 40 trees per acre to maximize the yield at 1,600 pounds per acre.

**Example 3.** A candy box is to be made out of a piece of cardboard that measures 8 by 12 inches. Squares of equal size will be cut out of each corner and the then all four sides will be folded up to form a rectangular box. What size square should be cut from each corner to obtain the maximum volume? What is the maximum volume?

$V\left(x\right)=x\left(8-2x\right)\left(12-2x\right)=4x^{3}-40x^{2}+96x 0\leq x\leq 4$

$V^{'}\left(x\right)=12x^{2}-80x+96=0$ at $x=\frac{10-2\sqrt{7}}{3}$

$V^{''}\left(x\right)=24x-80<0$ at $x=\frac{10-2\sqrt{7}}{3}$ implies concave down.

The cutout areas should be a square of size $\frac{10-2\sqrt{7}}{3}≈1.5695$ inches. The resulting volume is $\frac{640+448\sqrt{7}}{27}≈67.6036$ cubic inches.

**Example 4.** A freshwater pipeline is to be run from a source on the edge of a lake to a small resort community on an island 5 miles offshore.

A) If it costs 1.4 times as much to lay the pipe in the lake as it does on land, what should x be to minimize the total cost of the project? How long is the land section of the pipeline? How long is the section of pipeline under the water? If the cost to lay pipe on land is $\$50,000$ dollars per mile, what is the minimum cost (to the nearest thousand dollars)?

If c is the cost per mile to lay pipe on land, total cost can be written as

$$C\left(x\right)=c\left[\left(10-x\right)+1.4\sqrt{x^{2}+25}\right] 0\leq x\leq 10 $$

To minimize the total cost, we must minimize the quantity in square brackets (which I will call the pipeline cost factor):

$$f\left(x\right)=\left(10-x\right)+1.4\sqrt{x^{2}+25}$$

$$f^{'}\left(x\right)=-1+\frac{1.4x}{\sqrt{x^{2}+25}}=0 ⟹ x=\frac{25\sqrt{6}}{12} miles$$

$f^{''}\left(x\right)=\frac{35}{\left(x^{2}+25\right)^{{3}/{2}}}>0$ for $x>0$

The cost will be minimum when $x=\frac{25\sqrt{6}}{12}≈5.1031$ miles. The length of the pipeline on land is $10-\frac{25\sqrt{6}}{12}≈4.8969$ miles. The section under water is $\sqrt{\left(\frac{25\sqrt{6}}{12}\right)^{2}+25}≈7.1443$ miles long. The total cost for the project will be

$$C\left(x\right)=cf\left(x\right)=50,000\left[\left(10-\frac{25\sqrt{6}}{12}\right)+1.4\sqrt{\left(\frac{25\sqrt{6}}{12}\right)^{2}+25}\right]≈\$745,000.$$

B) Redo part A assuming it costs only 10% more to lay pipe in the lake as it does on land.$ $

$$f\left(x\right)=\left(10-x\right)+1.1\sqrt{x^{2}+25} 0\leq x\leq 10 $$

$f^{'}\left(x\right)=-1+\frac{1.1x}{\sqrt{x^{2}+25}}=0 ⟹ x=\frac{50\sqrt{21}}{21}≈10.9109$ miles

This is outside the domain of the function so there are no points with horizontal tangents. The minimum will be at one of the endpoints. Since $f'\left(x\right)$ is negative on the domain of the function, the minimum must occur at the right endpoint $\left(x=10\right)$. The length of the pipeline on land is $10$ miles. The section under water is $5$ miles long. The total cost for the project will be about $C\left(10\right)=cf\left(10\right)=50,000\left(1.1\sqrt{125}\right)≈\$615,000.$